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# Lab 11 & 12 Counting

## Objective

Solving exercises from the textbook in chapter 6.1-6.3.

## Current Lab Learning Outcomes (LLO)

By completion of the lab, the students should be able to understand:

1-Basics of counting.
2-Pigeonhole principle.
3-Permutations and combinations.

## Outline

Q1) Product rule in a and sum rule in b.

Q7) Product rule.

Q11) Product rule.

Q14)

Product rule for bit strings begin with three 0s,
then product rule for bit strings end with two 0s,
then Product rule for bit strings begin with three 0s and end with two 0s,
and then subtraction rule.

Q21) Pigeonhole principle.

Q15) General pigeonhole principle.

Q5) Combination.

Q6) Permutation.

**1- The Basics of Counting**

## Basic Counting Principles

We first present two basic counting principles, the **product rule** and the **sum rule**. Then we will show how they can be used to solve many different counting problems.

The product rule applies when a procedure is made up of separate tasks.

THE PRODUCT RULE Suppose that a procedure can be broken down into a sequence of two tasks. If there are $n_1$ ways to do the first task and for each of these ways of doing the first task, there are $n_2$ ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

THE SUM RULE If a task can be done either in one of $n_1$ ways or in one of $n_2$ ways, where none of the set of $n_1$ ways is the same as any of the set of $n_2$ ways, then there are $n_1 + n_2$ ways to do the task.

11-1) There are 18 mathematics majors and 325 computer science majors at a college.

a) In how many ways can two representatives be picked so that one is mathematics major and the other is a computer science major?

The procedure of picking the two representatives consists of picking a representative from mathematics majors, which can be done in 18ways, and then picking a representative from computer science major's which can be done in 325 ways. By using the product rule, there are 18*325 = 5850 ways can two representatives be picked.

b) In how many ways can one representative be picked who is either mathematics major or a computer science major?

There are 18 ways to choose a representative of the mathematics major and there are 325 ways to choose a representative who is a computer science major. Choosing a representative of the mathematics major is never the same as choosing a representative who is a mathematics major because no one is both a mathematics major and a computer science major. By the sum rule it follows that there are 18 + 325 = 343 possible ways to pick this representative.

If there are 5 who are both mathematics major and computer science major, then
By the subtraction rule 18+ 325-5 = 338.

The Subtraction Rule (Inclusion–Exclusion for Two Sets)

THE SUBTRACTION RULE If a task can be done in either $n_1$ ways or $n_2$ ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

11-7) A multiple-choice test contains 10 questions. There are four possible answers for each question.

a) In how many ways can a student answer the questions on the test if the student answers every question?

The procedure of a student answering the questions on the test consists of 10 tasks, namely, assigning to the question 1 one of the 4 answers, and then assigning to the second question one of the 4 answers, and so on up to the question 10. The product rule shows that there are 4.4.4.4.4.4.4.4.4.4 = 4^10 different ways that a student can answer the questions on the test.

b) In how many ways can a student answer the questions on the test if the student can leave answers blank? Same the (a) but here 5^10 (4 answers and one blank so 5 possibilities.)

11-11) How many bit strings of length ten both begin and end with a 1?

There are 1 * 2^8 * 1 = 256 ways to construct such a string. This follows by the product rule, because the first bit can be chosen in only one way, each of the second through the ninth bits can be chosen in two ways, and the last bit can be chosen in one way.

14-14) How many bit strings of length 10 either begin with three 0s or end with two 0s?

We can construct a bit string of length ten that begin with three 0s in 2^7 = 128 ways. This follows by the product rule, because the first three bits can be chosen in only one way and each of the other seven bits can be chosen in two ways. Similarly, we can construct a bit string of length ten ending with the two bits 00, in 2^8= 256 ways.
Some of the ways to construct a bit string of length ten that begin with three 0s are the same as the ways to construct a bit string of length ten ending with the two bits 00. There are 2^5= 32 ways to construct such a string. This follows by the product rule,

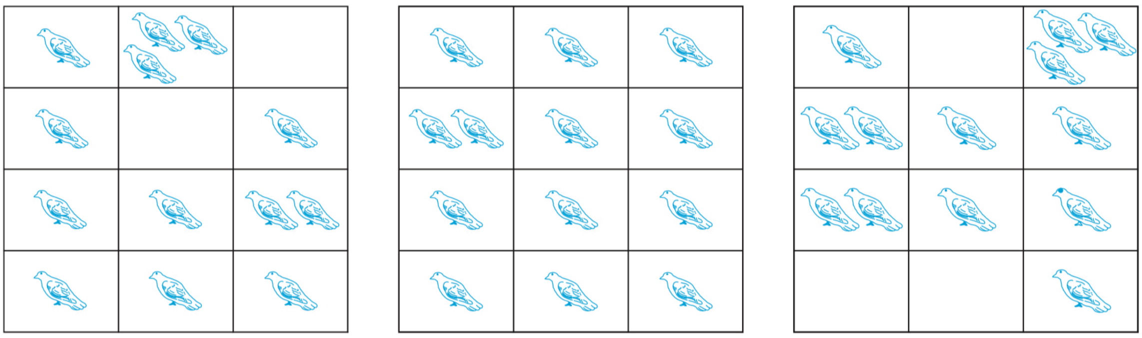
Consequently, the number of bit strings of length 10 either begin with three 0s or end with two 0s, which equals the number of ways to construct a bit strings of length 10 either begin with three 0s or end with two 0s,
equals 128 + 256 − 32 = 352.

## 2- The Pigeonhole Principle

### Introduction

Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost. Because there are 20 pigeons but only 19 pigeonholes, a least one of these 19 pigeonholes must have at least two pigeons in it. To see why this is true, note that if each pigeonhole had at most one pigeon in it, at most 19 pigeons, one per hole, could be accommodated. This illustrates a general principle called the **pigeonhole principle**, which states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it (see Figure 1). Of course, this principle applies to other objects besides pigeons and pigeonholes.

**THE PIGEONHOLE PRINCIPLE** If $k$ is a positive integer and $k+1$ or more objects are placed into $k$ boxes, then there is at least one box containing two or more of the objects.



(a) (b) (c)

**FIGURE 1** **There Are More Pigeons Than Pigeonholes.**

11-21) Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

K = Boxes = 0, 1, 2, 3 ← Pigeonholes

K+1 = Objects = Any five numbers ← Pigeons

Ex 3, 4, 5, 2, 1

| 0 | 1 |
|---|---|
| 4 | 1, 5 |
| **2** | **3** |
| 2 | 3 |

Because there are four possible remainders when an integer is divided by 4, the pigeonhole principle implies that given five integers, at least two have the same remainder.

# The Generalized Pigeonhole Principle

The pigeonhole principle states that there must be at least two objects in the same box when there are more objects than boxes. However, even more can be said when the number of objects exceeds a multiple of the number of boxes. For instance, among any set of 21 decimal digits there must be 3 that are the same. This follows because when 21 objects are distributed into 10 boxes, one box must have more than 2 objects.

**THE GENERALIZED PIGEONHOLE PRINCIPLE** If $N$ objects are placed into $k$ boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

11-15) There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?

imagie there are 2 times periods

pigeons and 2 different classes

how # rooms? 1

c | c pigeon-holes

if there are 3 classes

how # rooms? 2

pigeon-hole principle

if there are 10 classes

how # rooms? $\lceil \frac{10}{2} \rceil = 5$ rooms

generalized pigeon-hole principle.

if there are 13 classes

how # rooms? $\lceil \frac{13}{2} \rceil = 7$ rooms

**Solution:** The pigeon-holes are the time periods and the pigeons are the 677 different classes. By the generalized pigeon-hole principle there need to be at least $\lceil 677/38 \rceil$ classrooms to accomodate all classes.

## 3- Permutations and Combinations

If $n$ is a positive integer and $r$ is an integer with $1 \le r \le n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

$r$-permutations of a set with $n$ distinct elements.

If $n$ and $r$ are integers with $0 \le r \le n$, then $P(n, r) = \dfrac{n!}{(n-r)!}$.

The number of $r$-combinations of a set with $n$ elements, where $n$ is a nonnegative integer and $r$ is an integer with $0 \le r \le n$, equals

$$C(n, r) = \frac{n!}{r!\,(n-r)!}.$$

Let $n$ and $r$ be nonnegative integers with $r \le n$. Then $C(n, r) = C(n, n-r)$.

Let $S = \{1, 2, 3\}$. The ordered arrangement 3, 1, 2 is a permutation of $S$. The ordered arrangement 3, 2 is a 2-permutation of $S$.

Let $S$ be the set $\{1, 2, 3, 4\}$. Then $\{1, 3, 4\}$ is a 3-combination from $S$. (Note that $\{4, 1, 3\}$ is the same 3-combination as $\{1, 3, 4\}$, because the order in which the elements of a set are listed does not matter.)

The order matter in permutation. 3,1,2 is a permutation of S. 2,3,1 is another permutation. and 1,3,2 is another permutation

Combination is like sets, {1,3,4} is the same {4,1,3} is the same {3,4,1}
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The order is not matter in combinations.

12-10) Find the number of 5-permutations of a set with nine elements. P(9,5)

11-24) In how many ways can a set of five letters be selected from the English alphabet?

C(26,5)

11-5) A club has 25 members.

a) How many ways are there to choose four members of the club to serve on an executive committee?

C(25,4)= 25!/25!(25-4 !)

b) How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?

P(25,4)= 25!/ (25-4 !)

12-12) Seven women and nine men are on the faculty in the mathematics department at a school.

a) How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?

a) There are 16 faculty members in the department, therefore C(16, 5) different combinations of 5 people are possible. However, C(9, 5) of them consist entirely of men. Therefore there are C(16, 5) − C(9, 5) = 16!/ 5!11! − 9! /5!4! ways to select a committee with at least one women.

at least one woman

Either {M, M, M, M, W}
C(9, 4) * C(7, 1)
or {M, M, M, W, W}
+ C(9, 3) * C(7, 2)
or {M, M, W, W, W}
+ C(9, 2) * C(7, 3)
or {M, W, W, W, W}
+ C(9, 1) * C(7, 4)
or {W, W, W, W, W}
+ C(9, 0) * C(7, 5)

The product rule and sum rule are used with combination in solving this counting question.

11-3) How many permutations of {$a, b, c, d, e, f, g$} end with $a$? P(6,6)=6

11-4) How many possibilities are there for the win, place, and show (first, second, and third) positions in a horse race with 12 horses if all orders of finish are possible? P(12,3)

11-13) How many strings of eight English letters are there

a) That contain no vowels, if letters can be repeated? 26-5 = 21 letters, 21^8
b) That contain no vowels, if letters cannot be repeated? 21*20*19*18*17*16*15*14
c) That start with a vowel, if letters can be repeated? 5*26^7

11-18) A wired equivalent privacy (WEP) key for a wireless fidelity (WiFi) network is a string of either 10, 26, or 58 hexadecimal digits. How many different WEP keys are there? 16^10 + 16^26+ 16^58

11-19) Use the principle of inclusion–exclusion to find the number of positive integers less than 1,000,000 that are not divisible by either 4 or by 6. 666,667

11-23) How many bit strings of length 10 contain

**a)** exactly four 1s?
**b)** at most four 1s?
**c)** at least four 1s?
**d)** an equal number of 0s and 1s?

a) exactly four 1s? **We choose which four places contain those 1s : C(10, 4) = (10*9*8*7) / (4*3*2*1) = 210**
b) at most four 1s? **We add C(10, 4) + C(10, 3) + C(10, 2) + C(10, 1) + C(10, 0) = 386**
c) at least four 1s? **We subtract from the total number strings those that have only 0, 1, 2 or 3 1s :**
**$2^{10}$ – [ C(10, 0) + C(10, 1) + C(10, 2) + C(10, 3) ] = 1024 – 1 – 10 – 45 – 120 = 848**
d) an equal number of 0s and 1s? **Calculate C(10, 5) = 252**

11-26) How many permutations of the letters *ABCDEFG* contain

**a)** the string *BCD*?
**b)** the string *CFGA*?
**c)** the strings *BA* and *GF*?
**d)** the strings *ABC* and *DE*?
**e)** the strings *ABC* and *CDE*?
**f )**the strings *CBA* and *BED*?

a) the string BCD? **Count the permutations of A$EFG where $ represents BCD : 5*4*3*2*1 = 120**
b) the string CFGA? **Count the permuations of B&DE where & represents CFGA : 4*3*2*1 = 24**
c) the strings BA and GF? **Count the permutations of $CDE& where $ is BA and & is GF : 5! = 120**
d) the strings ABC and DE? **Count permuations of $&FG where $ is ABC and & is DE : 4! = 24**
e) the strings ABC and CDE? **Must contain ABCDE; count permutations of $FG : 3! = 6**
f) the strings CBA and BED? **It is not possible for B to be in the specified positions.**

# Rules and Theorems of Counting

| | Name | Rule or Theorem of Counting |
|---|---|---|
| Basics of counting | Product Rule | Suppose that a procedure can be broken down into a sequence of two tasks. If there are n1 ways to do the first task and for each of these ways of doing the first task, there are n2 ways to do the second task, then there are n1*n2 ways to do the procedure. |
| | Sum Rule | If a task can be done either in one of n1 ways or in one of n2 ways, where none of the set of n1 ways is the same as any of the set of n2 ways, then there are n1+ n2 ways to do the task. |
| | Subtraction Rule (Inclusion-Exclusion for Two Sets) | If a task can be done in either n1 ways or n2ways, then the number of ways to do the task is n1 + n2 minus the number of ways to do the task that are common to the two different ways. |
| Pigeonhole principle | Pigeonhole Principle | If k is a positive integer and k + 1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects. |
| | Generalized pigeonhole principle | If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects. |
| Permutations and combinations | Permutations | $P(n, r) = n(n-1)(n-2) \cdot\cdot\cdot (n-r+1)$<br>$P(n, r) = n! / (n-r)!$ |
| | Combinations | $C(n, r) = n! / r! (n-r)!$ |